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Multiscale asymptotic homogenization for multiphysics problems with multiple spatial and temporal scales: a coupled thermo-viscoelastic example problem

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Abstract

A systematic approach for analyzing multiple physical processes interacting at multiple spatial and temporal scales is developed. The proposed computational framework is applied to the coupled thermo-viscoelastic composites with microscopically periodic mechanical and thermal properties. A rapidly varying spatial and temporal scales are introduced to capture the effects of spatial and temporal fluctuations induced by spatial heterogeneities at diverse time scales. The initial-boundary value problem on the macroscale is derived by using the double scale asymptotic analysis in space and time. It is shown that an extra history-dependent long-term memory term introduced by the homogenization process in space and time can be obtained by solving a first order initial value problem. This is in contrast to the long-term memory term obtained by the classical spatial homogenization, which requires solutions of the initial-boundary value problem in the unit cell domain. The validity limits of the proposed spatial–temporal homogenized solution are established. Numerical example shows a good agreement between the proposed model and the reference solution obtained by using a finite element mesh with element size comparable to that of material heterogeneity.

Keywords: Multiphysics; Multiple scales; Composites; Homogenization

1. Introduction

The primary objective of the manuscript is to develop a systematic approach for analyzing multiple physical processes interacting at multiple spatial and temporal scales. The interacting physical processes may include mechanical, thermal, diffusion, chemical and electromagnetic fields. Most often these phenomena are treated as being uncoupled; hence, few separate analyses of the same system are typically performed for the complete prediction of the response. It is, however, understood that such treatments should be regarded as first order approximations to the real complex interactions.

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Table 1
Physical processes interaction matrix

Deformation (displacements)	Deformation induced heat production **E**	Strain controlled diffusion **C**	Strain controlled chemical activity **C**	Deformation induced electric and magnetic flux **E**
Thermal expansion **E**	Heat transfer (temperature)	Temperature controlled diffusion **C**	Temperature controlled chemical activity **C**	Temperature controlled electric and magnetic behavior **C**
Clustering, precipitation (eigenstrains) **E**	Diffusant concentration controlled heat transfer **C**	Diffusion (diffusant concentration)	Transport of reactants **E**	Diffusion induced electrical and magnetic potentials **E**
Reaction products accommodation (eigenstrains) **E**	Heat production during chemical reactions **E**	Transport of reaction products **E**	Chemical reactions	Reaction products induced electric and magnetic flux **E**
Deformation due to Lorentz force **E**	Heat input from Joule heating **E**	Transport of charged particles **E**	Electro-magnetic field controlled chemical activity **C**	Electro-magnetism

The coupling of mechanical, thermal, diffusion, chemical and electromagnetic fields (stress/strain, temperature, concentration, current) occurs through diverse phenomena, some of which are depicted in the interaction matrix shown in Table 1. The interaction matrix is "non-symmetric", with cell (i,j) representing the phenomenon induced by the process corresponding to field i and which influences field j. For instance, cell $(1,2)$ represents heating due to plastic deformation, while cell $(2,1)$ corresponds to thermal expansion and thermal stresses. A fully coupled analysis would consider all processes shown in the matrix, while fully uncoupled approach would only consider the diagonal entries. A one-way (or partially) coupled approach would consider a lower (or upper) triangular entries in the interaction matrix.

It is important to note that coupling of various physical processes in the mathematical model can be carried out either by considering additional terms in the field equations (equation coupling) or by allowing the constitutive law to depend on the interacting field (constitutive law coupling). An example of the first category is the effect of the temperature gradient on the stress field, which can be captured by adding a thermal stress term to equilibrium equations. A less familiar example within the same class of problems is the effect of diffusion (clustering) and chemical reactions (reaction products) on the stress field, which can be accounted for by the eigenstrain formulation (Fish and Belsky, 1995). An example from the second category is the influence of temperature upon diffusion which can be captured by simply considering the diffusion coefficients to be temperature dependent. Symbols E and C in Table 1 stand for the technique to be used for coupling the respective phenomenon, with E standing for "equation coupling" and C designating "constitutive law coupling".

Multiple length scales may exist in both space and time domains. To model spatial length scales induced by a spatial heterogeneity a small positive scaling parameter ε_l is introduced so that a local coordinates $\boldsymbol{y}$ can be identified and related to the global reference coordinates $\boldsymbol{x}$ by

$$\boldsymbol{y} = \boldsymbol{x}/\varepsilon_l \tag{1}$$

The response fields are then assumed to be the function of $(\boldsymbol{x},\boldsymbol{y})$ which represents the dependence on the local oscillations induced by the spatial heterogeneity in the vicinity of the macroscopic point. The value of ε_l and thus the validity of the spatial homogenization approach has been shown to depend on the following four factors (Fish and Belsky, 1995): (a) the size of the unit cell, (b) the volume fraction, (c) the mismatch of properties between micro-constituents, and (d) the macroscopic spatial gradients. The latter two factors imply that ε_l might be different for various physical processes.

To characterize the fast varying features of response fields in time domain, we assume that there exists a small positive scaling parameter ε_τ so that a fast time coordinate τ can be identified and defined as

$$\tau = t/\varepsilon_\tau \tag{2}$$

where t is a natural time coordinate. In contrast to the spatial scale separation, which is typically induced by a spatial heterogeneity, multiple time scales can be attributed to the following three factors (and their combinations):

1.1. Multiple time scales induced by the interaction of multiple physical processes

For illustration consider the stress-driven diffusion problem (Glicksman, 2000) in a one-dimensional periodic heterogeneous medium:

$$\text{mechanical field:} \quad \begin{array}{l} \sigma_{,x} = 0 \\ \sigma = E\{u_{,x} - \varsigma(C - C_0)\} \end{array} \tag{3}$$

$$\text{diffusion process:} \quad \dot{C} = \{D(C_{,x} - C\kappa\sigma_{,x})\}_{,x} \tag{4}$$

$$\text{BCs and IC:} \quad \begin{array}{l} u(x=0) = u_0 \sin(2\pi\omega t); \quad u(x=L) = 0 \\ C(t=0) = C_0 \end{array} \tag{5}$$

where σ denotes the stress; u the displacement field; C the concentration of solute; C_0 the initial concentration of solute; E the Young's modulus; ς the diffusion expansion coefficient; κ the material constant; D the diffusivity coefficient; and L the length of the model. The comma followed by a subscript variable denotes a partial derivative and superscribed dot stands for the time derivative. Material properties E, ς, D and κ take different values in distinct material phases.

Due to the linearity, the frequency of the mechanical response fields in (3) is the same as the loading frequency. For stress-driven diffusion process the diffusion driven force $DC\kappa\sigma_{,x}$ is oscillatory with frequency ω, while the diffusion rate is dominated by the diffusivity coefficient and material heterogeneity, which might be much slower than the oscillation of the driving force. The intrinsic diffusion time scale can be estimated by the time elapse t_r for a solute particle traveling throughout the unit cell. From (4), we have

$$t_r = \mathrm{O}\{\min(l^2/D)\} \tag{6}$$

where l is the length of the unit cell. Due to local periodicity, it follows from (6) that all the unbalance of concentration prior to the time elapse t_r is balanced throughout the model during t_r. Thus the time scaling parameter ε_τ is defined as the ratio between the period of mechanical oscillation $(1/\omega)$ and the intrinsic diffusion time scale t_r, i.e.

$$\varepsilon_\tau = 1/\omega t_r \tag{7}$$

Thus the second time scale is introduced due to the interaction of physical processes provided that the mechanical oscillation is at a significantly higher frequency than the diffusion process (i.e., $\varepsilon_\tau \ll 1$).

1.2. Multiple time scales induced by the difference between the frequency of the response fields and the material intrinsic time scale

As an illustrative example consider a rate-dependent material (Kelvin–Voigt type viscoelastic solid) under cyclic loading in one-dimension:

$$\begin{aligned} &\sigma_{,x} = 0 \\ &\sigma = Eu_{,x} - V\dot{u}_{,x} \end{aligned} \tag{8}$$

where V denotes viscosity. The boundary conditions are assumed to be the same as those defined in (5). The frequency of the mechanical response is ω due to the linearity, while the material intrinsic time scale is governed by $t_r = V/E$, which represents the rate of creep behavior (Yu and Fish, submitted for publication). The resulting time scaling parameter directly follows from (7).

1.3. Multiple time scales induced by multiple spatial scales

A typical example for this case is a dispersion phenomenon resulting from the wave propagation in heterogeneous media. The spatial scaling parameter ε_l is a consequence of disparity between the wave length and the unit cell size. For the details of this problem, we refer to Chen and Fish (in press).

In this paper attention is restricted to multiple time scales induced by the interaction of multiple physical processes. For demonstration purposes we consider an initial-boundary value problem for the thermo-viscoelastic composite. It consists of two spatial scales (micro-constituents and the macro-domain), two temporal scales (the time scale associated with an applied loading and the intrinsic time scale of the rate-dependent material) and two fully coupled physical processes (thermal and mechanical). Both the constitutive law coupling due to thermally sensitive material properties and the equation coupling induced by thermal stresses are taken into account for mechanical fields. For thermal fields, on the other hand, we assume that only the equation coupling occurs due to the mechanical dissipation and dilation effects.

To model the local oscillations of mechanical and thermal fields induced by spatial heterogeneities at diverse time scales, an asymptotic homogenization theory for multiple physical processes with multiple spatial and temporal scales is developed. When the loading is highly oscillatory in comparison with the material intrinsical time scale it is natural to incorporate a rapidly varying time scale in the asymptotic analysis. This fast time variable is defined to characterize the fast varying features of mechanical and thermal response fields in time domain.

Homogenization with multiple temporal scales could be traced back to Bensoussan et al. (1978) where the convergence analyses of the hyperbolic equations with oscillatory coefficients were established. Francfort (1983) generalized the conventional spatial homogenization method to the case of thermo-elastic composites. For the hyperbolic conservation law with rapid spatial fluctuations, Kevorkian and Bosley (1998) showed that the continuous initial data which is independent on the fast temporal scale may introduce a dependence on both fast spatial and temporal scales in the homogenized solutions. A handful of recent publications on this topic has been briefly reviewed in Chen and Fish (in press), where the role of multiple temporal scales in wave propagation in heterogeneous solids was investigated.

In Section 2 we start our presentation with a general setting of the initial-boundary value problem for the fully coupled Kelvin–Voigt thermo-viscoelastic composite. In addition to the usual space–time coordinates rapidly varying spatial and temporal scales are introduced to capture the effects of spatial, and temporal fluctuations. The macroscopic initial-boundary value problem is obtained by the double scale asymptotic analysis in space and time. It is shown that an extra long-term memory obtained from solving a first order initial value problem in macroscopic field is introduced into the homogenized solution. For the homogenization of viscoelastic heterogeneous media, it has been revealed that in addition to the original memories due to the viscosity of micro-constituents, an extra long-term fading memory is induced by the homogenization process. This phenomenon has been illustrated in Francfort and Suquet (1986), Galka et al. (1992), Glicksman (2000) and Iesan and Scalia (1996) for either Kelvin–Voigt or Maxwell viscoelastic model with only instantaneous memories in the micro-constituents. This extra long-term memory in the homogenized constitutive equation arises due to the interactions between fast spatial variation and time dependence of the coefficients of partial differential equations. We remark that all these studies were based

on the spatial homogenization only and the history-dependent integral kernel associated with the long-term memory is determined by a local initial-boundary value problem in the unit cell domain (Francfort and Suquet, 1986; Suquet, 1987). In this manuscript we show that the extra long-term memory term resulting from the homogenization process in space and time can be obtained by solving a first order initial value problem. This is in contrast to the long-term memory term obtained by the classical spatial homogenization, which requires solution of the initial-boundary value problem in the unit cell domain. This gives rise to an elegant homogenized solution which can be easily implemented into the numerical setting.

The asymptotic space–time homogenization formulation for the coupled Kelvin–Voigt thermo-viscoelastic composites is presented in Section 2. In addition to the usual space–time coordinates, rapidly varying spatial and temporal scales are introduced to capture the effects of spatial and temporal fluctuations. The macroscopic initial-boundary value problem is obtained by the double scale asymptotic analysis in space and time. Section 3 discusses various relations between the temporal and spatial scales as well as the validity of the proposed model. Numerical experiment comparing the proposed model with the classical spatial homogenization and the reference solution obtained by using a finite element mesh with element size comparable to that of material heterogeneity is given in Section 4.

2. Space–time multiple scale analysis for the coupled thermo-viscoelastic composite

In the present work, the Kelvin–Voigt viscoelastic model is considered for micro-constituents. In contrast to the multiscale analysis conducted by Boutin and Wong (1998), where a single frequency quasi-harmonic displacement field has been assumed so that the constitutive equation can be transformed to the elastic-like form, we consider a general setting of the coupled initial-boundary value problem. The distributed heat source arises due to the mechanical dissipation and the thermal dilation in micro-constituents. The thermally sensitive mechanical properties (stiffness and viscosity) as well as the thermal dilation term in the constitutive equation lead to the full coupling between mechanical response and thermal diffusion.

2.1. Definition of multiple spatial and temporal scales

The microstructure of a composite material is assumed to be locally periodic (Y-periodic) with a scale parameter ε_l defined by the representative volume element (RVE or unit cell). The macroscopic domain is represented by Ω^ε while Θ denotes the unit cell domain. We assume that RVE exists and its characteristic size l is small enough in comparison with the reference length l_r on the macroscale so that

$$\varepsilon_l = l/l_\mathrm{r}, \quad \varepsilon_l \ll 1 \tag{9}$$

In addition to the distinct spatial scales, we can identify at least two temporal scales in a typical thermo-viscoelastic problem: the time scale associated with the applied loading and the intrinsic time scale of the rate-dependent material. In the present work, we introduce a fast varying temporal coordinate τ to represent the fast oscillations of mechanical and thermal fields in time domain induced by the highly oscillatory loading. For linear systems and weakly non-linear systems, the characteristic length of τ is of the same order as the period τ_0 of loading profile (Boutin and Wong, 1998). We assume that the intrinsic time scale t_r, which is determined by material properties, describes a relatively long-term behavior, and thus the following relations hold:

$$\varepsilon_\tau = \tau_0/t_\mathrm{r}, \quad \varepsilon_\tau \ll 1 \tag{10}$$

where t_r is the characteristic length of the natural time scale denoted by t; ε_τ is the small scale parameter defined in the time domain. We start by considering the special case of

$$\varepsilon_l \approx \varepsilon_\tau \approx \varepsilon \tag{11}$$

Further discussion on the different relations between the temporal and spatial scales as well as on the validity of the homogenized solution is left to Section 4. It is important to note that the definitions of the multiple spatial and temporal length scales are physically distinct for the mechanical and thermal fields.

With the definition of the fast varying variables $\boldsymbol{y}$ and τ as well as the local Y-periodicity assumption, all the mechanical and thermal response quantities denoted by ϕ can be defined as

$$\phi(\boldsymbol{y},\tau) = \phi(\boldsymbol{y} + \boldsymbol{K}\hat{\boldsymbol{y}},\tau) \tag{12}$$

where $\hat{\boldsymbol{y}}$ is the basic period vector of the microstructure and $\boldsymbol{K}$ is a 3 by 3 diagonal matrix with arbitrary integer components. The corresponding εY-periodic function can be defined by using the conventional nomenclature:

$$\phi^\varepsilon(\boldsymbol{x},t) = \phi(\boldsymbol{x},\boldsymbol{y},t,\tau) \tag{13}$$

The differentiations with respect to space and time variables can be expressed using the chain rule:

$$\phi^\varepsilon_{,i} = \phi_{,x_i} + \varepsilon^{-1}\phi_{,y_i} \quad \text{and} \quad \dot{\phi}^\varepsilon = \phi_{,t} + \varepsilon^{-1}\phi_{,\tau} \tag{14}$$

where the comma followed by a subscript variable denotes a partial derivative and superscribed dot denotes the time derivative. Summation convention for repeated subscripts is adopted except for the subscripts x and y.

2.2. Initial-boundary value problem statement for the coupled thermo-viscoelastic composites

Attention is restricted to small deformations and small temperature increases. The microscopic constituents are assumed to be homogeneous and their thermo-viscoelastic behavior can be described by the following initial-boundary value problem in the macroscopic domain Ω^ε (Francfort and Suquet, 1986; Iesan and Scalia, 1996).

(1) Equation of motion

$$\rho^\varepsilon \ddot{u}^\varepsilon_i = \sigma^\varepsilon_{ij,j} + b_i \tag{15}$$

where ρ^ε is the density; $\ddot{u}^\varepsilon_i$ and σ^ε_{ij} the displacement and stress components, respectively; b_i the body force component assumed to be independent of the fast varying coordinates.

(2) Constitutive equation

$$\sigma^\varepsilon_{ij} = L^\varepsilon_{ijkl} e^\varepsilon_{kl} + V^\varepsilon_{ijkl}\dot{e}^\varepsilon_{kl} - \beta^\varepsilon_{ij}\theta^\varepsilon \tag{16}$$

where e^ε_{kl} and $\dot{e}^\varepsilon_{kl}$ denote the strain and the strain rate components, respectively; θ^ε the temperature change from the initial temperature; L^ε_{ijkl} and V^ε_{ijkl} the elastic stiffness and the viscosity tensor components, respectively; $\beta^\varepsilon_{ij} = L^\varepsilon_{ijkl}\alpha^\varepsilon_{kl}$ where α^ε_{kl} denotes the coefficient of thermal expansion and $\beta^\varepsilon_{ij}\theta^\varepsilon$ stands for the components of the thermal stress. We assume that the fourth-rank tensor components L^ε_{ijkl} and V^ε_{ijkl} as well as the second-rank tensor components α^ε_{kl} and β^ε_{kl} satisfy conditions of symmetry and positivity. We further assume that L^ε_{ijkl} and V^ε_{ijkl} are thermally sensitive which lead to the constitutive law coupling between the mechanical and thermal fields, while α^ε_{kl} is assumed to be insensitive to the temperature change, i.e. $\alpha^\varepsilon_{ij} \equiv \alpha_{ij}(\boldsymbol{x},\boldsymbol{y})$

(3) Kinematic equation

$$e^\varepsilon_{ij} = \tfrac{1}{2}(u^\varepsilon_{i,j} + u^\varepsilon_{j,i}) \tag{17}$$

(4) Energy equation

$$\lambda^{\varepsilon}\dot{\theta}^{\varepsilon} = q^{\varepsilon}_{i,i} + Q + V^{\varepsilon}_{ijkl}\dot{e}^{\varepsilon}_{ij}\dot{e}^{\varepsilon}_{kl} - T_0\beta^{\varepsilon}_{ij}\dot{e}^{\varepsilon}_{ij} \tag{18}$$

where $\lambda^{\varepsilon} = \rho^{\varepsilon}c^{\varepsilon}$ and c^{ε} is the specific heat per unit mass; q^{ε}_i denotes heat flux; T_0 the initial temperature and Q the heat supply. Both T_0 and Q are assumed to be independent of the fast varying coordinates. The total temperature T^{ε} is given by $T^{\varepsilon} = T_0 + \theta^{\varepsilon}$. The mechanical dissipation term $V^{\varepsilon}_{ijkl}\dot{e}^{\varepsilon}_{ij}\dot{e}^{\varepsilon}_{kl}$ and the dilation induced heat supply $T_0\beta^{\varepsilon}_{ij}\dot{e}^{\varepsilon}_{ij}$ fall into the category of equation coupling for the thermal fields.

(5) Linear thermal diffusion

$$q^{\varepsilon}_i = k^{\varepsilon}_{ij}\theta^{\varepsilon}_{,j} \tag{19}$$

where k^{ε}_{ij} denotes the thermal conductivity tensor components assumed to be symmetric and positive definite. Since k^{ε}_{ij} is insensitive to temperature changes, it is also assumed to be time-independent, i.e., $k^{\varepsilon}_{ij} \equiv k_{ij}(\boldsymbol{x}, \boldsymbol{y})$.

(6) Initial and boundary conditions

The non-oscillatory initial conditions at $t = 0$ and $\tau = 0$ are imposed on both time scales, i.e., the initial state is assumed to be spatially and temporally smooth (Francfort, 1983). The thermo-mechanical boundary conditions are also assumed to be non-oscillatory and the interfaces between different microscopic constituents are perfectly bonded.

2.3. Double scale asymptotic analysis in space and time

To solve for the initial-boundary value problem described in Section 2.2, we start by introducing the following double scale asymptotic expansions:

$$u^{\varepsilon}_i = \sum_{m=0,1,\ldots} \varepsilon^m u^m_i(\boldsymbol{x}, \boldsymbol{y}, t, \tau); \qquad \theta^{\varepsilon} = \sum_{m=0,1,\ldots} \varepsilon^m \theta^m(\boldsymbol{x}, \boldsymbol{y}, t, \tau) \tag{20}$$

where u^m_i and θ^m are Y-periodic functions and m denotes the order of the associated component in the expansion. According to (20) and the chain rule in (14), the asymptotic expansions of strain (17) and the strain rate can be expressed as

$$e^{\varepsilon}_{ij} = \sum_{m=-1,0,\ldots} \varepsilon^m e^m_{ij}(\boldsymbol{x}, \boldsymbol{y}, t, \tau); \qquad \dot{e}^{\varepsilon}_{ij} = \sum_{m=-2,-1,\ldots} \varepsilon^m \dot{e}^m_{ij}(\boldsymbol{x}, \boldsymbol{y}, t, \tau) \tag{21}$$

where, with the definition of symmetric displacement gradients

$$e^n_{ijx} = \tfrac{1}{2}(u^n_{i,x_j} + u^n_{j,x_i}) \quad \text{and} \quad e^n_{ijy} = \tfrac{1}{2}(u^n_{i,y_j} + u^n_{j,y_i}), \quad n = 0, 1, 2, \ldots \tag{22}$$

the strain and strain rate components for various orders of ε in (21) are given as

$$e^{-1}_{ij}(\boldsymbol{x}, \boldsymbol{y}, t, \tau) = e^0_{ijy}, \qquad e^n_{ij}(\boldsymbol{x}, \boldsymbol{y}, t, \tau) = e^n_{ijx} + e^{n+1}_{ijy}, \quad n = 0, 1, 2, \ldots \tag{23}$$

and

$$\begin{aligned} &\dot{e}^{-2}_{ij}(\boldsymbol{x}, \boldsymbol{y}, t, \tau) = e^0_{ijy,\tau}, \qquad \dot{e}^{-1}_{ij}(\boldsymbol{x}, \boldsymbol{y}, t, \tau) = e^0_{ijy,t} + (e^0_{ijx} + e^1_{ijy})_{,\tau} \\ &\dot{e}^n_{ij}(\boldsymbol{x}, \boldsymbol{y}, t, \tau) = (e^n_{ijx} + e^{n+1}_{ijy})_{,t} + (e^{n+1}_{ijx} + e^{n+2}_{ijy})_{,\tau}, \quad n = 0, 1, 2, \ldots \end{aligned} \tag{24}$$

Consequently, the expansion of the stress field is obtained by substituting the expansions in (22) into the constitutive equation (16), which gives

$$\sigma^{\varepsilon}_{ij} = \sum_{m=-2,-1,\ldots} \varepsilon^m \sigma^m_{ij}(\boldsymbol{x}, \boldsymbol{y}, t, \tau) \tag{25}$$

where

$$\sigma_{ij}^{-2}(\boldsymbol{x},\boldsymbol{y},t,\tau) = V_{ijkl}^{\varepsilon}\dot{\boldsymbol{e}}_{kl}^{-2}, \qquad \sigma_{ij}^{-1}(\boldsymbol{x},\boldsymbol{y},t,\tau) = L_{ijkl}^{\varepsilon}e_{kl}^{-1} + V_{ijkl}^{\varepsilon}\dot{\boldsymbol{e}}_{kl}^{-1}$$
$$\sigma_{ij}^{n}(\boldsymbol{x},\boldsymbol{y},t,\tau) = L_{ijkl}^{\varepsilon}e_{kl}^{n} - \beta_{ij}^{\varepsilon}\theta^{n} + V_{ijkl}^{\varepsilon}\dot{\boldsymbol{e}}_{kl}^{n}, \quad n = 0,1,2,\ldots \tag{26}$$

Similarly, the expansion of the heat flux is obtained by using (19) and (20) such that

$$q_i^{\varepsilon} = \sum_{m=-1,0,\ldots} \varepsilon^m q_i^m(\boldsymbol{x},\boldsymbol{y},t,\tau) \tag{27}$$

where

$$q_i^{-1}(\boldsymbol{x},\boldsymbol{y},t,\tau) = k_{ij}^{\varepsilon}\theta_{,y_j}^{0} \quad \text{and} \quad q_i^{n}(\boldsymbol{x},\boldsymbol{y},t,\tau) = k_{ij}^{\varepsilon}(\theta_{,x_j}^{n} + \theta_{,y_j}^{n+1}), \quad n = 0,1,2,\ldots \tag{28}$$

Having defined the asymptotic expansions for the mechanical and thermal fields, the equation of motion (15) and the energy equation (18) can be stated in terms of two sets of equations with increasing order of ε starting from $O(\varepsilon^{-4})$ for the energy equation and $O(\varepsilon^{-3})$ for the equation of motion. Solving these equations successively yields the $O(\varepsilon^{0})$ initial-boundary value problem and the homogenized constitutive equations.

2.3.1. $O(\varepsilon^{-4})$ and $O(\varepsilon^{-3})$ equations

We first consider the $O(\varepsilon^{-4})$ energy equation

$$V_{ijkl}^{\varepsilon}\dot{\boldsymbol{e}}_{ij}^{-2}\dot{\boldsymbol{e}}_{kl}^{-2} = 0 \tag{29}$$

Due to symmetry and positivity of the viscosity tensor V_{ijkl}^{ε} as illustrated by Eq. (29), along with Eqs. (24) and (26), leads to

$$\dot{\boldsymbol{e}}_{ij}^{-2} = e_{ijy,\tau}^{0} = 0; \qquad \sigma_{ij}^{-2} = 0 \tag{30}$$

Since the initial conditions are non-oscillatory it implies $e_{ijy}^{0}(\boldsymbol{x},t,\tau=0) = 0$ and the first equation in (30) gives $e_{ijy}^{0} = 0$, i.e., u_i^0 is independent of $\boldsymbol{y}$

$$u_i^0 \equiv u_i^0(\boldsymbol{x},t,\tau) \tag{31}$$

With (30), it can be easily shown that the $O(\varepsilon^{-3})$ order equation of motion and energy equation are automatically satisfied.

2.3.2. $O(\varepsilon^{-2})$ equation

The $O(\varepsilon^{-2})$ order equations take the following form:

$$\rho^{\varepsilon}u_{i,\tau\tau}^{0} = \sigma_{ij,y_j}^{-1}$$
$$q_{i,y_i}^{-1} + V_{ijkl}^{\varepsilon}\dot{\boldsymbol{e}}_{ij}^{-1}\dot{\boldsymbol{e}}_{kl}^{-1} = 0 \tag{32}$$

Integrating the first equation in (32) over the unit cell domain and making use of the Y-periodicity of σ_{ij}^{-1} and the Y-independence of u_i^0 as shown in (31), as well as the non-oscillatory initial conditions, yields

$$u_{i,\tau\tau}^{0} = 0 \Rightarrow u_i^0 \equiv u_i^0(x,t) \tag{33}$$

Apparently, u_i^0 is independent of fast spatial and temporal variables and thus it represents the macroscopic displacement field while its symmetrical gradient e_{ijx}^{0} represents the macroscopic strain field. With this in mind along with (24) and (26), the $O(\varepsilon^{-2})$ order equation of motion in (32) is reduced to

$$\sigma_{ij,y_j}^{-1} = (V_{ijkl}^{\varepsilon}e_{kly,\tau}^{1})_{,y_j} = 0 \tag{34}$$

which can be further reduced to $e^1_{kly,\tau}=0$ due to the fact that V^ε_{ijkl} is symmetric and positive definite and $u^1_{i,\tau}$ is Y-periodic. Due to the non-oscillatory initial conditions it follows that

$$e^1_{ijy}=0 \Rightarrow u^1_i \equiv u^1_i(\boldsymbol{x},t,\tau) \tag{35}$$

indicating that u^1_i is independent of $\boldsymbol{y}$. Based on (33) and (34), the following identities for the strain and stress fields can be identified:

$$e^{-1}_{ij}=0; \quad \dot{\boldsymbol{e}}^{-1}_{ij}=0; \quad \sigma^{-1}_{ij}=0 \tag{36}$$

Thus the first non-vanishing terms in the expansion of strain, strain rate and stress fields are all $O(\varepsilon^0)$ order.

As for the $O(\varepsilon^{-2})$ order energy equation in (32), the simplified form can be obtained by exploiting (28) and (36) which yields

$$(k^\varepsilon_{ij}\theta^0_{,y_j})_{,y_i}=0 \tag{37}$$

Once again, the symmetry and positivity of k^ε_{ij} as well as the Y-periodicity of θ^0 provides the solution of (37) in the form of

$$\theta^0_{,y_j}=0 \Rightarrow \theta^0 \equiv \theta^0(\boldsymbol{x},t,\tau) \quad \text{and} \quad q^{-1}_i=0 \tag{38}$$

Therefore the first non-vanishing term in the expansion of heat flux q^ε_i also starts from $O(\varepsilon^0)$ order.

2.3.3. $O(\varepsilon^{-1})$ equation

With the solutions obtained from the lower order equations, (36) and (38), the $O(\varepsilon^{-1})$ order equations take the following form:

$$\begin{aligned} \rho^\varepsilon u^1_{i,\tau\tau} &= \sigma^0_{ij,y_j} \\ \lambda^\varepsilon \theta^0_{,\tau} &= q^0_{i,y_i} \end{aligned} \tag{39}$$

We first consider the $O(\varepsilon^{-1})$ order energy equation. Averaging it over the unit cell domain and utilizing the Y-periodicity of q^0_i and Y-independence of θ^0, we have

$$\theta^0_{,\tau}=0 \Rightarrow \theta^0 \equiv \theta^0(\boldsymbol{x},t) \tag{40}$$

Thus, θ^0 is independent of the fast varying variables and can be viewed as the macroscopic temperature true change.

Along with (28) and (40), $O(\varepsilon^{-1})$ order energy equation turns into

$$\left\{k^\varepsilon_{ij}(\theta^0_{,x_j}+\theta^1_{,y_j})\right\}_{,y_i}=0 \tag{41}$$

where, as we assumed in (19), k^ε_{ij} is independent of temperature change, i.e., $k^\varepsilon_{ij} \equiv k_{ij}(\boldsymbol{x},\boldsymbol{y})$. Due to the linearity of (41) (θ^0 is independent of $\boldsymbol{y}$), the solution of θ^1 can be expressed by the following decomposition

$$\theta^1(\boldsymbol{x},\boldsymbol{y},t,\tau)=\mu_i(\boldsymbol{y})\theta^0_{,x_j}+P(\boldsymbol{x},t,\tau) \tag{42}$$

where $P(\boldsymbol{x},t,\tau)$ is an arbitrary Y-independent function and $\mu_i(\boldsymbol{y})$ is determined by

$$\left\{k^\varepsilon_{ij}(\delta_{jm}+\mu_{m,y_j}(\boldsymbol{y}))\right\}_{,y_i}=0 \quad \text{in } \Theta \tag{43}$$

where δ_{jm} is Kronecker delta. Eq. (43) along with the corresponding periodic boundary conditions represents a typical linear unit cell problem. Finite element method can be used to solve for this equation (see, for example, Sanchez-Palencia, 1980). Upon the solutions of (43) and (42), $O(\varepsilon^0)$ order heat flux defined in (28) can be written as

$$q_i^0 = k_{ij}^{\varepsilon}(\delta_{jm} + \mu_{m,y_j}(\boldsymbol{y}))\theta^0_{,x_m} \tag{44}$$

We now consider $O(\varepsilon^{-1})$ order equation of motion in (39). Using the same considerations as for Eq. (32) and recalling that u_i^1 is independent of y, the $O(\varepsilon^{-1})$ order equation of motion provides

$$u^1_{i,\tau\tau} = 0 \Rightarrow u_i^1 \equiv u_i^1(\boldsymbol{x}, t) \tag{45}$$

It can be seen that u_i^1 is independent of the fast varying variables. Consequently, the $O(\varepsilon^{-1})$ order equation of motion, combined with definitions in (24) and (26) as well as the relations in (33) and (40), turns into

$$\sigma^0_{ij,y_j} = \left\{ L^{\varepsilon}_{ijkl} e^0_{klx} + V^{\varepsilon}_{ijkl}(e^0_{klx,t} + e^2_{kly,\tau}) - \beta^{\varepsilon}_{ij}\theta^0 \right\}_{,y_j} = 0 \tag{46}$$

According to (33) and (40), we conclude that e^0_{klx}, $e^0_{klx,t}$ and θ^0 represent the macroscopic strain, strain rate and temperature fields which are independent of $\boldsymbol{y}$ and τ. To solve (46) for $e^2_{kly,\tau}$ in terms of these macroscopic response quantities, we recall that L^{ε}_{ijkl} and V^{ε}_{ijkl} have been assumed to be thermally sensitive in the present work. Following Boutin and Wong (1998) and noting that our attention is restricted to small temperature changes in comparison with the initial temperature T_0, the thermally sensitive material properties are defined as

$$\begin{aligned} L^{\varepsilon}_{ijkl} &\approx L^{\mathrm{ini}}_{ijkl}(\boldsymbol{x}, \boldsymbol{y}) + \theta^0(\boldsymbol{x}, t) A^{\mathrm{ini}}_{ijkl}(\boldsymbol{x}, \boldsymbol{y}) \\ V^{\varepsilon}_{ijkl} &\approx V^{\mathrm{ini}}_{ijkl}(\boldsymbol{x}, \boldsymbol{y}) + \theta^0(\boldsymbol{x}, t) B^{\mathrm{ini}}_{ijkl}(\boldsymbol{x}, \boldsymbol{y}) \end{aligned} \tag{47}$$

where L^{ini}_{ijkl} and V^{ini}_{ijkl} are elastic stiffness and viscosity tensor components evaluated at the initial temperature; A^{ini}_{ijkl} and B^{ini}_{ijkl} represent the thermal sensitivity tensor components associated with the elastic stiffness and viscosity evaluated at the initial temperature, respectively.

As stated in Section 2.1 the thermal expansion coefficient $\alpha^{\varepsilon}_{kl}$ is assumed to be independent of temperature change, i.e., $\alpha^{\varepsilon}_{kl} \equiv \alpha^{\varepsilon}_{kl}(\boldsymbol{x}, \boldsymbol{y})$. Since $\beta^{\varepsilon}_{ij} = L^{\varepsilon}_{ijkl}\alpha^{\varepsilon}_{kl}$ all the quantities in (46), except for $e^2_{kly,\tau}$, are known to be independent of τ. It follows that $e^2_{kly,\tau}$ should be also independent of τ. Following Fish and Belsky (1995) and considering the temperature change θ^0 as a state variable determined by the energy equation (evolution law) we introduce the following decomposition

$$u^2_{k,\tau}(\boldsymbol{x}, \boldsymbol{y}, t, \tau) = \varsigma_{kmn}(\boldsymbol{y})(e^0_{mnx} + w^0_{mn}) + \vartheta_{kmn}(\boldsymbol{y})(e^0_{mnx,t} + w^0_{mn,t}) + U_k(\boldsymbol{x}, t, \tau) \tag{48}$$

where $\varsigma_{kmn}(\boldsymbol{y})$ and $\vartheta_{kmn}(\boldsymbol{y})$ are the third-rank tensors which are symmetric with respect to m, n; $w^0_{mn} \equiv w^0_{mn}(\boldsymbol{x}, t)$ and $w^0_{mn,t} \equiv w^0_{mn,t}(\boldsymbol{x}, t)$ represent the temperature induced macroscopic strain and strain rate (eigenstrain and eigenstrain rate), respectively; $U_k(\boldsymbol{x}, t, \tau)$ is an arbitrary function. The symmetric gradient of (48) is given as

$$e^2_{kly,\tau}(\boldsymbol{x}, \boldsymbol{y}, t) = \psi_{klmn}(\boldsymbol{y})(e^0_{mnx} + w^0_{mn}) + \chi_{klmn}(\boldsymbol{y})(e^0_{mnx,t} + w^0_{mn,t}) \tag{49}$$

where $\psi_{klmn}(\boldsymbol{y})$ and $\chi_{klmn}(\boldsymbol{y})$ denote the symmetric gradient of $\varsigma_{kmn}(\boldsymbol{y})$ and $\vartheta_{kmn}(\boldsymbol{y})$ with respect to $\boldsymbol{y}$, i.e.

$$\begin{aligned} \psi_{klmn} &= \{\varsigma_{kmn_{,y_l}} + \varsigma_{lmn_{,y_k}}\}/2 \\ \chi_{klmn} &= \{\vartheta_{kmn_{,y_l}} + \vartheta_{lmn_{,y_k}}\}/2 \end{aligned} \tag{50}$$

To solve for $\varsigma_{kmn}(\boldsymbol{y})$ and $\vartheta_{kmn}(\boldsymbol{y})$ as well as for the temperature change induced macroscopic response w^0_{mn} and $d^0_{mn,t}$, we start by substituting (49) into (46), which yields

$$\left\{ (L^{\varepsilon}_{ijmn} + V^{\varepsilon}_{ijkl}\psi_{klmn})e^0_{mnx} + V^{\varepsilon}_{ijkl}\psi_{klmn}w^0_{mn} - \beta^{\varepsilon}_{ij}\theta^0 + V^{\varepsilon}_{ijkl}(\delta_{km}\delta_{ln} + \chi_{klmn})e^0_{mnx,t} + V^{\varepsilon}_{ijkl}\chi_{klmn}w^0_{mn,t} \right\}_{,y_j} = 0 \tag{51}$$

Since ψ_{klmn} and χ_{klmn} are both assumed to be independent of time and thus temperature change, these two local concentration functions can be evaluated by the following two linear unit cell problems obtained from (51) when $t = 0$ so that $\theta^0 = 0$ and $w^0_{mn} = w^0_{mn,t} = 0$:

$$\left\{L^{\text{ini}}_{ijmn} + V^{\text{ini}}_{ijkl}\psi_{klmn}\right\}_{,y_j} = 0 \quad \text{in } \Theta \tag{52}$$

$$\left\{V^{\text{ini}}_{ijkl}(\delta_{km}\delta_{ln} + \chi_{klmn})\right\}_{,y_j} = 0 \quad \text{in } \Theta \tag{53}$$

Again, similarly to (43), the above two linear unit cell problems along with the periodic boundary conditions can be solved for $\varsigma_{kmn}(\boldsymbol{y})$ and $\vartheta_{kmn}(\boldsymbol{y})$, respectively, by using finite element method. Owing to the Y-periodicity of L^{ini}_{ijkl} and V^{ini}_{ijkl}, both ψ_{klmn} and χ_{klmn} are also Y-periodic. It can be readily proved that ψ_{klmn} and χ_{klmn} are polarization functions whose integrations over the unit cell domain vanish due to the periodicity. The temperature change induced macroscopic response fields, w^0_{mn} and $w^0_{mn,t}$, are obtained by multiplying (51) with the periodic function $\vartheta_{kmn}(\boldsymbol{y})$ and then integrating it by parts over the unit cell domain, which leads to the initial value problem for w^0_{mn}:

$$\bar{\boldsymbol{a}}_{pqmn}(\boldsymbol{x},t)w^0_{mn,t} + \bar{\boldsymbol{b}}_{pqmn}(\boldsymbol{x},t)w^0_{mn} = -\bar{\boldsymbol{f}}_{pqmn}(\boldsymbol{x},t)e^0_{mnx} - \bar{\boldsymbol{g}}_{pqmn}(\boldsymbol{x},t)e^0_{mnx,t} + \bar{\boldsymbol{h}}_{pq}(\boldsymbol{x},t)\theta^0 \tag{54}$$

with the initial condition: $w^0_{mn} = 0$ and $t = 0$. The time-dependent coefficients are given as

$$\begin{aligned}
\bar{\boldsymbol{a}}_{pqmn} &= \langle \chi_{ijpq} V^{\varepsilon}_{ijkl} \chi_{klmn} \rangle \\
\bar{\boldsymbol{b}}_{pqmn} &= \langle \chi_{ijpq} V^{\varepsilon}_{ijkl} \psi_{klmn} \rangle \\
\bar{\boldsymbol{f}}_{pqmn} &= \langle \chi_{ijpq} (L^{\varepsilon}_{ijmn} + V^{\varepsilon}_{ijkl} \psi_{klmn}) \rangle \\
\bar{\boldsymbol{g}}_{pqmn} &= \langle \chi_{ijpq} V^{\varepsilon}_{ijkl} (\delta_{km}\delta_{ln} + \chi_{klmn}) \rangle \\
\bar{\boldsymbol{h}}_{pq} &= \langle \chi_{ijpq} \beta^{\varepsilon}_{ij} \rangle
\end{aligned} \tag{55}$$

and the spatial averaging operator $\langle \bullet \rangle$ is defined as

$$\langle \bullet \rangle = \frac{1}{|\Theta|} \int_{\Theta} \bullet \, \mathrm{d}\Theta \tag{56}$$

where $|\Theta|$ is the volume of the unit cell. The solution of w^0_{mn} from (54) is a history-dependent function which leads to the long-term fading memory in the macroscopic constitutive equation. In contrast to the long-term memory induced by the classical spatial homogenization process (Francfort and Suquet, 1986; Sanchez-Palencia, 1980; Yi et al., 1998), which involves solving an initial-boundary value problem in the unit-cell domain (see (A.16) in Appendix A), the present long-term memory is obtained by solving the first order initial value problem at each Gauss-point in the macro-domain.

To this end, (49) can be expressed in a concise form as

$$e^2_{kly,\tau}(\boldsymbol{x},\boldsymbol{y},t) = \xi_{klmn}(\boldsymbol{x},\boldsymbol{y},t)e^0_{mnx} + \zeta_{klmn}(\boldsymbol{x},\boldsymbol{y},t)e^0_{mnx,t} - \eta_{kl}(\boldsymbol{x},\boldsymbol{y},t)\theta^0 + \gamma_{klmn}(\boldsymbol{x},\boldsymbol{y},t)w^0_{mn} \tag{57}$$

where

$$\begin{aligned}
\xi_{klmn} &= \psi_{klmn} - \chi_{klpq}(\bar{\boldsymbol{a}}_{pqij})^{-1}\bar{\boldsymbol{f}}_{ijmn} \\
\zeta_{klmn} &= \chi_{klmn} - \chi_{klpq}(\bar{\boldsymbol{a}}_{pqij})^{-1}\bar{\boldsymbol{g}}_{ijmn} \\
\eta_{kl} &= -\chi_{klpq}(\bar{\boldsymbol{a}}_{pqij})^{-1}\bar{\boldsymbol{h}}_{ij} \\
\gamma_{klmn} &= \psi_{klmn} - \chi_{klpq}(\bar{\boldsymbol{a}}_{pqij})^{-1}\bar{\boldsymbol{b}}_{ijmn}
\end{aligned} \tag{58}$$

Finally, we remark that the time-dependence of these four parameters is due to the temperature-dependence of L^{ε}_{ijkl} and V^{ε}_{ijkl}.

2.3.4. $O(\varepsilon^0)$ equation

The $O(\varepsilon^0)$ order equations of motion and energy, along with (33) and (36), can be written as

$$\begin{aligned}\rho^{\varepsilon}\ddot{u}^0_i &= \sigma^0_{ij,x_j} + \sigma^1_{ij,y_j} + b_i \\ \lambda^{\varepsilon}\dot{\theta}^0 &= q^0_{i,x_i} + q^1_{i,y_i} + Q + V^{\varepsilon}_{ijkl}\dot{e}^0_{ij}\dot{e}^0_{kl} - T_0\beta^{\varepsilon}_{ij}\dot{e}^0_{ij}\end{aligned} \tag{59}$$

where the $O(\varepsilon^0)$ order stress σ^0_{ij} and strain rate $\dot{e}^0_{ij}$ can be obtained from (24), (26), (45) and (57) which gives

$$\begin{aligned}\sigma^0_{ij} &= L^{\varepsilon}_{ijkl}e^0_{kl} + V^{\varepsilon}_{ijkl}\dot{e}^0_{kl} - \beta^{\varepsilon}_{ij}\theta^0 \\ \dot{e}^0_{ij} &= \xi_{ijmn}e^0_{mnx} + (\delta_{im}\delta_{jn} + \zeta_{ijmn})e^0_{mnx,t} - \eta_{ij}\theta^0 + \gamma_{ijmn}w^0_{mn}\end{aligned} \tag{60}$$

It can be seen that both σ^0_{ij} and $\dot{e}^0_{ij}$ are independent of τ.

For the $O(\varepsilon^0)$ order equation of motion in (59), the volume average over the unit cell domain provides

$$\bar{\rho}\ddot{u}^0_i = \bar{\sigma}^0_{ij,x_j} + b_i \tag{61}$$

where the macroscopic stress is determined by the homogenized constitutive equation

$$\bar{\sigma}^0_{ij} = \overline{L}_{ijkl}e^0_{klx} + \overline{V}_{ijkl}e^0_{klx,t} - \bar{\beta}_{ij}\theta^0 + \overline{\Lambda}_{ijkl}w^0_{kl} \tag{62}$$

and the homogenized coefficients are given by

$$\begin{aligned}\bar{\rho} &= \langle\rho^{\varepsilon}\rangle \\ \overline{L}_{ijkl} &= \langle L^{\varepsilon}_{ijkl} + V^{\varepsilon}_{ijmn}\xi_{mnkl}\rangle \\ \overline{V}_{ijkl} &= \langle V^{\varepsilon}_{ijmn}(\delta_{mk}\delta_{nl} + \zeta_{mnkl})\rangle \\ \bar{\beta}_{ij} &= \langle\beta^{\varepsilon}_{ij} + V^{\varepsilon}_{ijkl}\eta_{kl}\rangle \\ \overline{\Lambda}_{ijkl} &= \langle V^{\varepsilon}_{ijmn}\gamma_{mnkl}\rangle\end{aligned} \tag{63}$$

The $O(\varepsilon^0)$ homogenized energy equation is obtained by averaging the second equation in (59) over the unit cell domain and making use of (40) and (44)

$$\bar{\lambda}\dot{\theta}^0 = (\bar{k}_{ij}\theta^0_{,x_j})_{,x_i} + Q + \langle V^{\varepsilon}_{ijkl}\dot{e}^0_{ij}\dot{e}^0_{kl}\rangle - T_0\langle\beta^{\varepsilon}_{ij}\dot{e}^0_{ij}\rangle \tag{64}$$

where $\dot{e}^0_{ij}$ is given in (60); $\bar{k}_{ij}$ is the homogenized thermal conductivity given by

$$\bar{k}_{ij} = \langle k^{\varepsilon}_{im}(\delta_{jm} + \mu_{j,y_m})\rangle \tag{65}$$

and the average volumetric specific heat $\bar{\lambda}$ is defined as

$$\bar{\lambda} = \langle\rho^{\varepsilon}c^{\varepsilon}\rangle \tag{66}$$

Finally, we remark that the initial and boundary conditions for the components in asymptotic expansions are defined to satisfy the imposed conditions of the source problem defined in Section 2.2. Thus the initial

and boundary conditions for the $O(\varepsilon^0)$ components coincide with those imposed on the source problem, while the components of higher order of ε follow trivial initial and boundary conditions.

3. Validity of the homogenized solutions

In the previous section we have shown that the extra long-term memory in the homogenized constitutive equation (62) can be obtained by solving a first order initial value problem in the macroscopic domain. Therefore, the proposed homogenized constitutive model (62) offers significant computational advantages by eliminating the need for evaluating the local initial-boundary problem associated with the long-term memory in homogenization processes (Francfort and Suquet, 1986; Yi et al., 1998).

Despite the remarkable simplicity of the present mode, it is important to investigate under what circumstances the fast temporal scale exists and the homogenized solution is valid. Recall that the homogenized solution obtained in Section 2.3 is based on the assumption $\varepsilon_l \approx \varepsilon_\tau \approx \varepsilon$. We have investigated other relations between the temporal and spatial scales and results of these findings are summarized in Table 2. The derivation details corresponding to various combinations of multiple spatial and temporal scales are presented in the Appendix. It is shown that once the fast temporal scale in mechanical fields exists, the present solution is unconditionally valid; the multiple temporal scales in thermal fields have no effect on the $O(\varepsilon^0)$ homogenized solutions. On the other hand, when the fast temporal scale does not exist in mechanical fields, our approach is not valid and the homogenized solutions take the classical form obtained by the spatial homogenization only (Francfort and Suquet, 1986; Sanchez-Palencia, 1980).

The existence of fast varying temporal scale is determined by the characteristic temporal length of response fields and material itself. The intrinsic temporal scale can be estimated by requiring the two major terms in the equation of motion (15), i.e. elastic and viscous contributions, to be of the same order, which yields

$$t_r = O\{\min(\|V^\varepsilon_{ijkl}\|/\|L^\varepsilon_{ijkl}\|)\} \tag{67}$$

where $\|\bullet\|$ means the norm of $\bullet$; t_r is the characteristic length of macroscopic reference time scale. Physically, the ratio defined in (67) characterizes the rate of creep behavior. Note that the thermal dilation effect is typically very small and inertial force is assumed to be not in dominance.

For the energy equation, we denote the characteristic length of the intrinsic temporal scale in thermal fields as $\hat{t}_r$. To quantify $\hat{t}_r$, we require the first two terms in the energy equation (18), i.e. the specific heat increase and thermal conductivity term, are of the same order, which yields

$$\hat{l}_{adv} = \min\{\hat{t}_r\|k^\varepsilon_{ij}\|/\|\lambda^\varepsilon\|\}^{1/2} \tag{68}$$

where $\hat{l}_{adv}$ represents the heat front advance during the time elapse $\hat{t}_r$. Then, we can infer by such reasoning that, when $\hat{l}_{adv} \geqslant l$, the heat front has swept through the unit cell during $\hat{t}_r$ so that all the heat fluctuations generated before the start of $\hat{t}_r$ has been smoothed out in the unit cell, and thus the homogenized thermal

Table 2
Validity of the homogenized solutions ($m, n = 1, 2, 3, \ldots; \varepsilon \ll 1$)

Thermal fields	Mechanical fields		
	$\varepsilon_l = \varepsilon^m$, $\varepsilon_\tau = \varepsilon$	$\varepsilon_l = \varepsilon$, $\varepsilon_\tau = \varepsilon^m$	$\varepsilon_l = \varepsilon^m$, $\tau_0/t_r \geqslant 1$
$\varepsilon_l = \varepsilon^n$, $\varepsilon_\tau = \varepsilon$	Valid	Valid	Invalid
$\varepsilon_l = \varepsilon$, $\varepsilon_\tau = \varepsilon^n$	Valid	Valid	Invalid
$\varepsilon_l = \varepsilon^n$, $\tau_0/t_r \geqslant 1$	Valid	Valid	Invalid

fields could be reached provided that no more heat is generated during $\hat{t}_r$. In this sense, $\hat{t}_r$ can be approximated as

$$\hat{t}_r \approx \mathrm{O}\left(l^2/\min\{\|k_{ij}^{\varepsilon}\|/\|\lambda^{\varepsilon}\|\}\right) \tag{69}$$

4. Numerical examples

In this section, we first study the thermo-mechanical behavior of a one-dimensional biphasic bar. Reference solutions are obtained by using a very fine mesh, whose grid size is chosen to resolve the features of the microstructure. The classical spatial homogenization solution (Francfort and Suquet, 1986) is also computed and compared to the present spatial–temporal homogenization. As a second example, we consider a simple three-dimensional model. In both numerical examples, the inertia force is set to be very small in comparison with the elastic and viscous effects so that the quasi-static problems could be considered.

4.1. One-dimensional example

Consider a biphasic bar with the periodic local structures as shown in Fig. 1. The cross-sectional area of the bar is assumed to be unity. The volume fractions of two phases are denoted by d_1 for phase Θ_1 and d_2 for phase Θ_2, such that $d_1 + d_2 = 1$ and $\Theta_1 \cup \Theta_2 = \Theta$. In the one-dimensional example, $|\,\Theta\,| = l$ where l is the length of the unit cell. Material properties of each phase are denoted as follows: Young's modulus at initial temperature E^{ini}, thermal sensitivity of Young's modulus A^{ini}, viscosity at initial temperature V^{ini}, thermal sensitivity of viscosity B^{ini}, thermal expansion coefficient α, thermal conductivity k, density ρ, and specific heat per unit mass c. Thermal sensitivity of material properties and the thermal expansion effects are both neglected. The following material properties are selected:

Geometry: $L = 200$ mm, $l = 1$ mm, $d_1 = 0.4$, $d_2 = 0.6$

Phase 1: $E_1^{\mathrm{ini}} = 40$ GPa, $V_1^{\mathrm{ini}} = 80$ GPa s, $A_1^{\mathrm{ini}} = 0$, $B_1^{\mathrm{ini}} = 0$, $\alpha_1 = 0$, $\rho_1 = 2200$ kg/m^3, $c_1 = 1500$ J/kg K, $k_1 = 50$ W/m K

Phase 2: $E_2^{\mathrm{ini}} = 4$ GPa, $V_2^{\mathrm{ini}} = 20$ GPa s, $A_2^{\mathrm{ini}} = 0$, $B_2^{\mathrm{ini}} = 0$, $\alpha_2 = 0$, $\rho_2 = 1100$ kg/m^3, $c_2 = 3000$ J/kg K, $k_2 = 10$ W/m K

The adiabatic boundary conditions are imposed on the bar so that no heat transfer occurs between the bar and the ambience. Also, we eliminate the body force b_1 and the input heat supply Q so that mechanical dissipation is the only heat source. We further assume that the initial temperature $T_0 = 300$ K is uniform throughout the bar. The bar is subjected to the cyclic displacement boundary condition given as

$$u(x = 0, t) = 0.5\bar{u}(1 - \cos 2\pi\omega t) \tag{70}$$

where $\bar{u}$ and ω are the loading parameters.

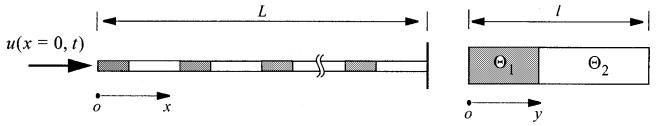

Fig. 1. One-dimensional bar and the associated unit cell.

With the above definitions, three linear unit cell problems defined in (43), (52) and (53) could be solved analytically by imposing periodic Dirichlet boundary conditions on the unit cell domain. The results are summarized below:

$$\begin{aligned}
&\mu_1 = \frac{d_2(k_2-k_1)}{d_1k_2+d_2k_1}y; \quad \psi_1 = \frac{d_2(E_2^{\text{ini}}-E_1^{\text{ini}})}{d_1E_2^{\text{ini}}+d_2E_1^{\text{ini}}}y; \quad \chi_1 = \frac{d_2(V_2^{\text{ini}}-V_1^{\text{ini}})}{d_1V_2^{\text{ini}}+d_2V_1^{\text{ini}}}y \\
&\mu_2 = \frac{d_1(k_1-k_2)}{d_1k_2+d_2k_1}(y-l); \quad \psi_2 = \frac{d_1(E_1^{\text{ini}}-E_2^{\text{ini}})}{d_1E_2^{\text{ini}}+d_2E_1^{\text{ini}}}(y-l); \quad \chi_2 = \frac{d_1(V_1^{\text{ini}}-V_2^{\text{ini}})}{d_1V_2^{\text{ini}}+d_2V_1^{\text{ini}}}(y-l)
\end{aligned} \tag{71}$$

The homogenized mechanical and thermal field equations for the one-dimensional example are given as

$$\begin{aligned}
&\bar{\sigma}^0_{11,x} = 0 \\
\text{mechanical fields:} \quad &\bar{\sigma}^0_{11} = \overline{E}e^0_{11} - \bar{\beta}\theta^0 + \overline{V}\dot{e}^0_{11} + \overline{\Lambda}w^0_{11} \\
&\bar{a}w^0_{11,t} + \bar{b}w^0_{11} = -\bar{f}e^0_{11} - \bar{g}e^0_{11,t} + \bar{h}\theta^0; \quad w^0_{11}(t=0) = 0
\end{aligned} \tag{72}$$

$$\text{thermal fields:} \quad \bar{\lambda}\theta^0_{,t} = (\bar{k}\theta^0_{,x})_{,x} + \frac{1}{l}\int_0^l (V\dot{e}^0_{11}\dot{e}^0_{11} - T_0\beta\dot{e}^0_{11})\mathrm{d}y \tag{73}$$

where the homogenized coefficients in one-dimensional case, i.e. $\overline{E}$, $\bar{\beta}$, $\overline{V}$, $\overline{\Lambda}$ and $\bar{\lambda}$, are given in (63) and (66); the one-dimensional local parameters ξ, ζ, η and γ in (63) and (66) can be obtained from (58) along with the solution (71) for the unit cell problems; one-dimensional local average parameters $\bar{a}$, $\bar{b}$, $\bar{f}$, $\bar{g}$ and $\bar{h}$ are given in (55).

The reference time scale t_{r} for the mechanical and $\hat{t}_{\text{r}}$ for thermal fields can be evaluated by (67) and (69) respectively, which yields $t_{\text{r}} = \text{O}(1)$ s and $\hat{t}_{\text{r}} = \text{O}(10^{-1})$ s. The response of the bar under one-cycle loading, i.e. $\omega t \in [0,1]$ in (70) with $\bar{u} = 0.04$ mm and $\omega = 0.1$, 1, and 10 rad/s, are shown in Figs. 2–4. The reference solutions are obtained by using very fine mesh in the heterogeneous model and then taking volume average over the unit cell domain. In Fig. 2 we consider $\omega = 0.1$ rad/s which gives $\tau_0 = 10$ s and thus $\varepsilon_\tau > 1$ for both mechanical and thermal fields. It can be seen that in this case the present formulation errs badly while the solution obtained by the classical spatial homogenization, which needs the existence of ε_l only, is in good agreement with the reference solution. Fig. 3 illustrates a mixed case, where $\tau_0 = 1$ s and thus $\varepsilon_\tau = \text{O}(10^{-1})$ for the mechanical field, while $\varepsilon_\tau > 1$ for the thermal field. The numerical results show that the present space–time homogenization provides a good approximation to the reference solutions. This is consistent with the observations made in Section 3 and Table 2. In Fig. 3, τ_0 is further reduced to 0.1 s and in this case we have $\varepsilon_\tau = \text{O}(10^{-2})$ in the mechanical field and $\varepsilon_\tau = \text{O}(10^{-1})$ in the thermal field. Excellent agreement between the reference solutions and the present formulation can be observed.

4.2. Three-dimensional example

We consider a single ply of unidirectionally reinforced fibrous composite subjected to the uniform pressure as shown in Fig. 5. Fibers are assumed to be aligned in X direction and the ply is supported by a rigid foundation. We further assume that the thickness (10 mm in Z direction) of the ply is very small compared with the size in other two dimensions (X and Y directions). Thus the thermo-mechanical response of the composite ply can be described by a stack of unit cells along Z direction subjected to the periodic boundary conditions in X and Y directions. The finite element model of the unit cell is shown in Fig. 5.

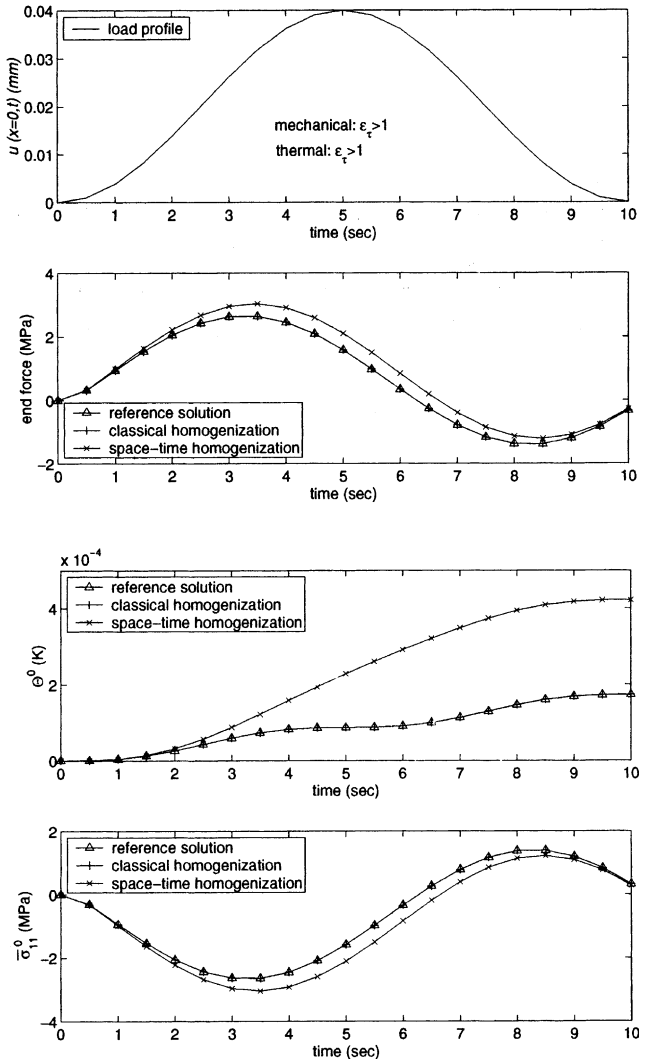

Fig. 2. Loading profile and mechanical/thermal responses for the one-dimensional example ($\omega = 10$ rad/s, 0.1 Hz).

We assume that each phase in the unit cell is isotropic and homogeneous. The material properties are summarized below:

$$\text{Fiber:}\quad E_1^{\text{ini}} = 37.92\ \text{GPa},\ \ v_1^{\text{ini}} = 0.21,\ \ V_1^{\text{ini}} = 6\ \text{GPa s},\ \ \alpha_1 = 1\times 10^{-6}\ \text{K}^{-1},\ \ \rho_1 = 2200\ \text{kg/m}^3,$$
$$c_1 = 1\ \text{kJ/kg K},\ \ k_1 = 0.2\ \text{W/m K},\ \ d_1 = 0.267$$

$$\text{Matrix:}\quad E_2^{\text{ini}} = 6.89\ \text{GPa},\ \ v_2^{\text{ini}} = 0.33,\ \ V_2^{\text{ini}} = 3\ \text{GPa s},\ \ \alpha_2 = 5\times 10^{-5}\ \text{K}^{-1},\ \ \rho_2 = 1100\ \text{kg/m}^3,$$
$$c_2 = 2\ \text{kJ/kg K},\ \ k_2 = 1\ \text{W/m K},\ \ d_2 = 0.733$$

where v^{ini} is Poisson's ratio at initial temperature, we assume that the thermal sensitivities for elastic stiffness and viscosity are: $A_{ijkl}^{\text{ini}} = -0.01 L_{ijkl}^{\text{ini}}$ and $B_{ijkl}^{\text{ini}} = -0.05 V_{ijkl}^{\text{ini}}$. The adiabatic boundary conditions are imposed

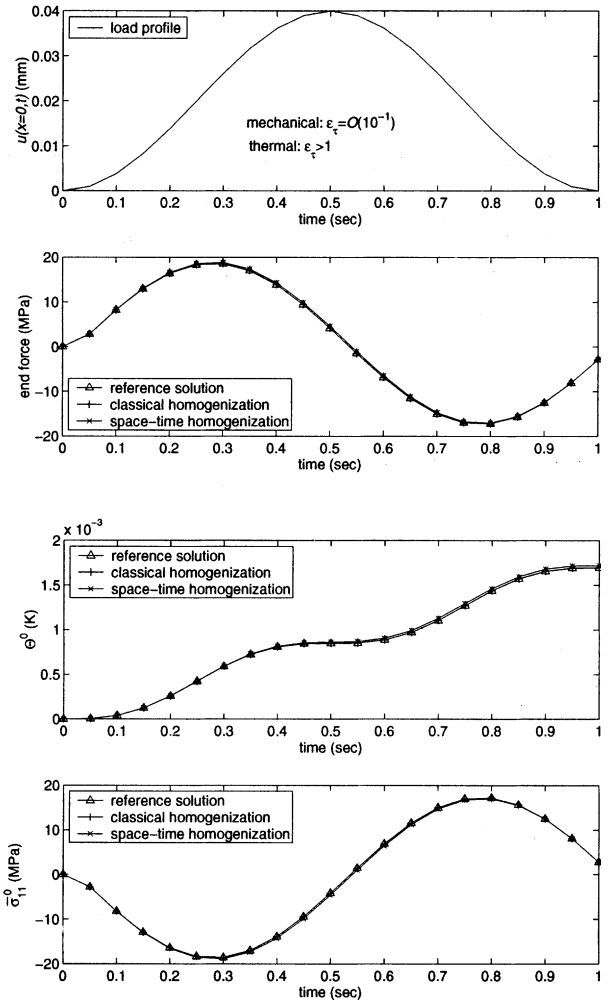

Fig. 3. Loading profile and mechanical/thermal responses for the one-dimensional example ($\omega = 1$ rad/s, 1 Hz).

on the ply, and the body force b_i and the heat supply Q are set to zero. The initial temperature $T_0 = 300$ K is assumed to be uniform throughout the ply. As shown in Fig. 5, the displacement boundary condition in the form of

$$u(x = 0, t) = 0.5\bar{u}(1 - \cos 2\pi\omega t) \tag{74}$$

is applied in Z direction on the free boundary, where $\bar{u}$ and ω are the loading parameters.

The reference solution obtained by deploying a very fine mesh in the three-dimensional strip is compared against the homogenized solution. The simulation results of one-cycle 10-Hz loading are plotted in Figs. 6 and 7, where the loading history and the corresponding comparisons between the reference solutions and the homogenized solutions of end force, temperature change and stress component are illustrated. It can be seen that the dilation effect offsets the temperature increase in the unloading phase. In Fig. 8, the distribution of the residual stress σ^0_{33} in the unit cell right after the one-cycle loading, which is reconstructed from the homogenized solution (60), compares well with the reference solution.

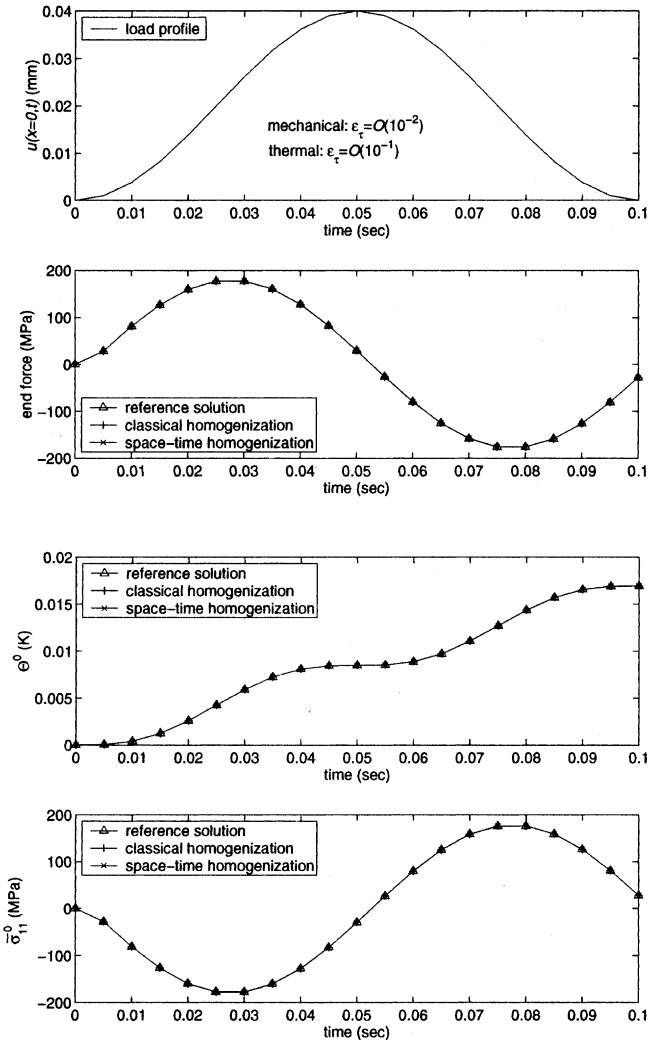

Fig. 4. Loading profile and mechanical/thermal responses for the one-dimensional example ($\omega = 0.1$ rad/s, 10 Hz).

5. Summary and future research directions

A multiscale space–time asymptotic homogenization procedure for analyzing multiple physical processes interacting at multiple spatial and temporal scales is developed and applied to the coupled thermo-visco-elastic composites. Rapidly varying spatial and temporal scales are introduced to capture the oscillations induced by local heterogeneities at diverse time scales. The homogenized initial-boundary value problem along with the homogenized constitutive equations are derived using the double scale asymptotic analysis in space and time. It is shown that the additional long-term memory induced by homogenization process can be obtained by solving a first order initial value problem as opposed to solving initial-boundary value problem in the unit cell domain in the case of the classical spatial homogenization.

We have identified two diverse time scales resulting from the input excitations and rate dependent material behavior. Further investigation reveals higher order terms in the asymptotic expansions grow

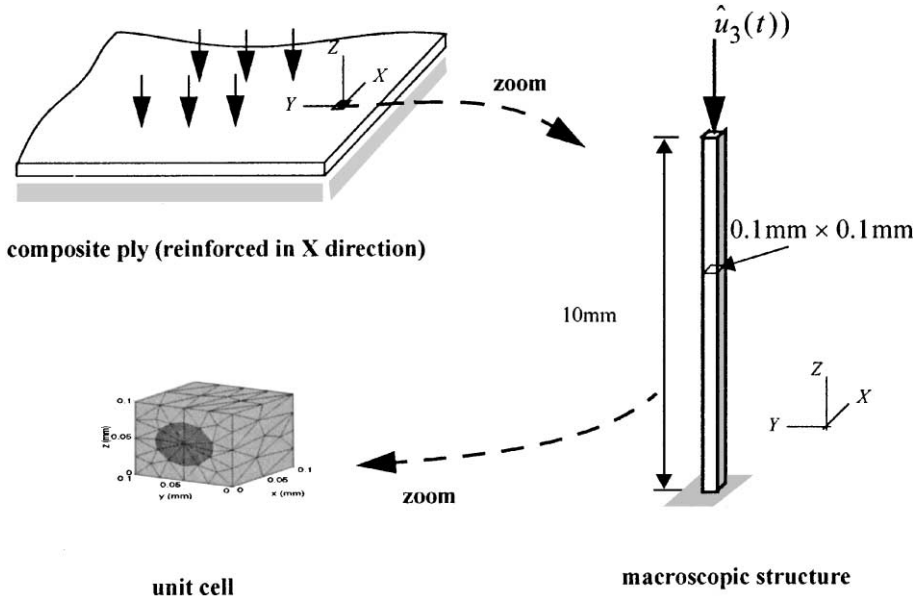

Fig. 5. Composite plate and the associated unit cell.

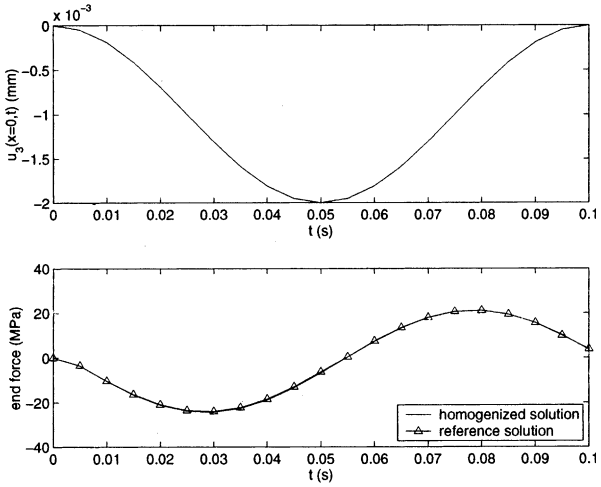

Fig. 6. Loading history and the end force obtained with the homogenized and reference solutions.

unbounded in time and will affect the accuracy of the first order homogenized solutions when the observation time window is long enough. A regularization scheme to suppress this secular time dependence has been recently proposed for wave propagation problems in elastic heterogeneous solid (Chen and Fish, in press). The applicability of this regularization scheme for the present model will be investigated in the future work.

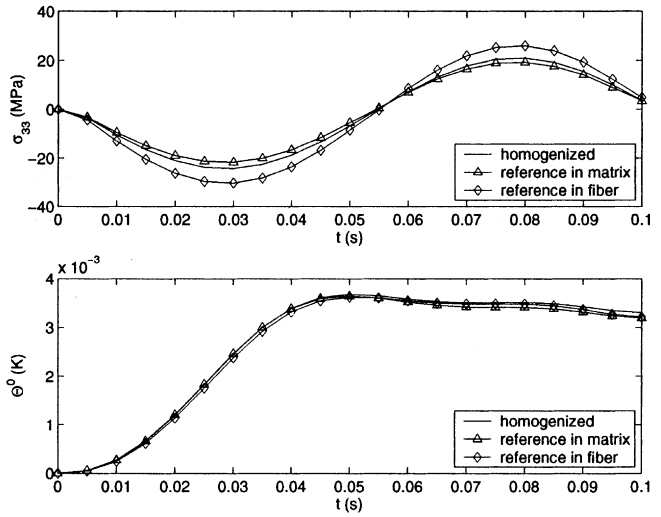

Fig. 7. Temperature and stress σ_{33}^0 obtained with the homogenized and reference solutions.

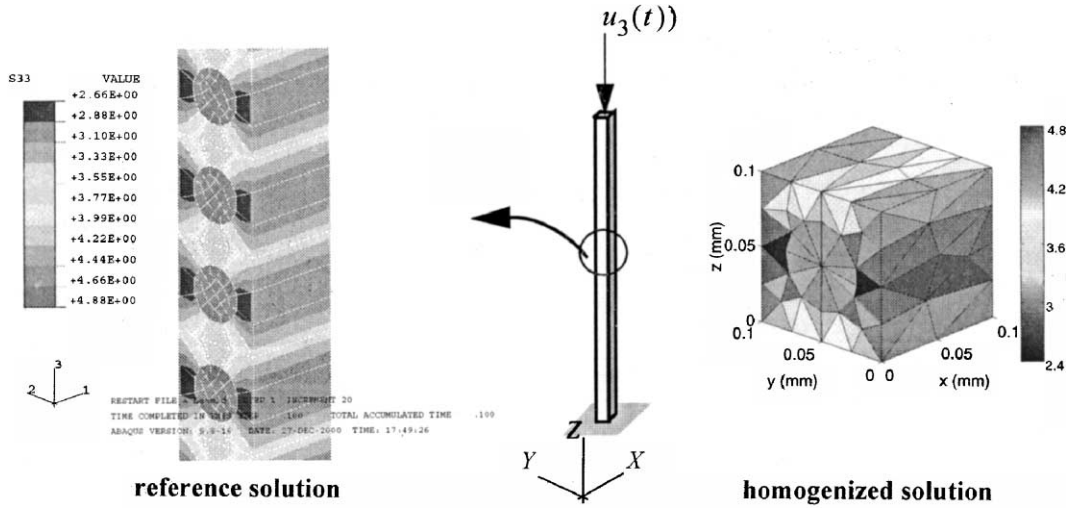

Fig. 8. Distribution of the local residual stress σ_{33}^0 right after the loading.

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Appendix A

In this section, we summarize the results of $O(\varepsilon^0)$ homogenized solutions corresponding to different combinations of spatial and temporal length scales as shown in Table 2. We start by introducing the av-

eraging operator in the time domain with fast temporal oscillating variable τ. According to Francfort (1983), it requires

$$\lim_{\hat{\tau}\to\infty}\frac{1}{\hat{\tau}}\int_0^{\tau} f(\tau)\mathrm{d}\tau \qquad \text{exist and be finite} \tag{A.1}$$

In the Laplace transformed domain, (A.1) is equivalent to

$$\lim_{s\to 0} s\tilde{f}(s) \qquad \text{exist and be finite} \tag{A.2}$$

where s is the variable in the transformed domain corresponding to τ; $\tilde{f}(s)$ is Laplace transformation of function $f(\tau)$.

A.1. Thermal fields: $\varepsilon_l = \varepsilon^n$, $\varepsilon_\tau = \varepsilon$

A.1.1. Mechanical fields: $\varepsilon_l = \varepsilon^m$, $\varepsilon_\tau = \varepsilon$

The $\mathrm{O}(\varepsilon^0)$ order equation of motion and energy equation can be expressed as

$$\begin{aligned}\rho^{\varepsilon}\ddot{\boldsymbol{u}}_i^0 &= \sigma_{ij,x_j}^0 + \sigma_{ij,y_j}^m + b_i\\ \lambda^{\varepsilon}(\dot{\theta}^0 + \theta_{,\tau}^1) &= q_{i,x_i}^0 + q_{i,y_i}^1 + V_{ijkl}^{\varepsilon}\dot{\boldsymbol{e}}_{ij}^0\dot{\boldsymbol{e}}_{kl}^0 - T_0\beta_{ij}^{\varepsilon}\dot{\boldsymbol{e}}_{ij}^0\end{aligned} \tag{A.3}$$

where u_i^0 and θ^0 are both independent of the fast varying variables and represent the macroscopic displacement and temperature change; the $\mathrm{O}(\varepsilon^0)$ order stress σ_{ij}^0 and strain rate $\dot{\boldsymbol{e}}_{kl}^0$ take the following form

$$\begin{aligned}\sigma_{ij}^0 &= L_{ijkl}^{\varepsilon}e_{klx}^0 + V_{ijkl}^{\varepsilon}\dot{\boldsymbol{e}}_{kl}^0 - \beta_{ij}^{\varepsilon}\theta^0\\ \dot{\boldsymbol{e}}_{kl}^0 &= e_{klx,t}^0 + e_{kly,\tau}^{m+1}\end{aligned} \tag{A.4}$$

where $e_{kly,\tau}^{m+1}$ is obtained by solving a linear unit cell problem with periodic boundary conditions

$$\left\{L_{ijkl}^{\varepsilon}e_{klx}^0 + V_{ijkl}^{\varepsilon}(e_{klx,t}^0 + e_{kly,\tau}^{m+1}) - \beta_{ij}^{\varepsilon}\theta^0\right\}_{,y_j} = 0 \tag{A.5}$$

Following (46) and (57), the solution of (A.5) is given as

$$e_{kly,\tau}^{m+1}(\boldsymbol{x},\boldsymbol{y},t) = \xi_{klmn}(\boldsymbol{x},\boldsymbol{y},t)e_{mnx}^0 + \zeta_{klmn}(\boldsymbol{x},\boldsymbol{y},t)e_{mnx,t}^0 - \eta_{kl}(\boldsymbol{x},\boldsymbol{y},t)\theta^0 + \gamma_{klmn}(\boldsymbol{x},\boldsymbol{y},t)w_{mn}^0 \tag{A.6}$$

where the four time-dependent parameters ξ_{klmn}, ζ_{klmn}, η_{kl} and γ_{klmn} are defined in (58). To this end, we conclude that the present local constitutive equation (A.4) is identical to (60). Making use of the homogenization process for $\mathrm{O}(\varepsilon^0)$ equation of motion in Section 2.3.4 leads to the same homogenized equation of motion:

$$\bar{\boldsymbol{\rho}}\ddot{\boldsymbol{u}}_i^0 = \bar{\boldsymbol{\sigma}}_{ij,x_j}^0 + b_i \tag{A.7}$$

and the associated homogenized constitutive equation

$$\bar{\boldsymbol{\sigma}}_{ij}^0 = \bar{\boldsymbol{L}}_{ijkl}e_{klx}^0 + \overline{\boldsymbol{V}}_{ijkl}e_{klx,t}^0 - \bar{\boldsymbol{\beta}}_{ij}\theta^0 + \overline{\boldsymbol{\Lambda}}_{ijkl}w_{kl}^0 \tag{A.8}$$

where the homogenized coefficients in (A.8) are defined in (63).

For the $\mathrm{O}(\varepsilon^0)$ energy equation, the heat flux q_i^0 is determined by

$$q_i^0 = k_{ij}^{\varepsilon}(\theta_{,x_j}^0 + \theta_{,y_j}^n) \tag{A.9}$$

where $\theta_{,y_j}^n$ is obtained by solving a linear unit cell problem

$$\left\{k_{ij}^{\varepsilon}(\theta^0_{,x_j}+\theta^n_{,y_j})\right\}_{,y_i}=0 \tag{A.10}$$

Following (41)–(44), we have

$$\theta^n_{,y_j}(\boldsymbol{x},\boldsymbol{y},t)=\mu_{i,y_j}(\boldsymbol{y})\theta^0_{,x_i} \quad \text{and} \quad q_i^0=k_{ij}^{\varepsilon}(\delta_{jk}+\mu_{k,y_j}(\boldsymbol{y}))\theta^0_{,x_k} \tag{A.11}$$

where $\mu_i(\boldsymbol{y})$ is defined in (43). The homogenized energy equation is obtained by applying the spatial and temporal averaging processes, (56) and (A.2), to the second equation in (A.3) and making use of (A.11) so that

$$\bar{\lambda}\dot{\theta}^0=(\bar{k}_{ij}\theta^0_{,x_j})_{,x_i}+Q+\langle V_{ijkl}^{\varepsilon}\dot{e}_{ij}^0\dot{e}_{kl}^0\rangle-T_0\langle\beta_{ij}^{\varepsilon}\dot{e}_{ij}^0\rangle \tag{A.12}$$

where all the quantities have the same definitions as those in (64).

A.1.2. Mechanical fields: $\varepsilon_l=\varepsilon$, $\varepsilon_\tau=\varepsilon^m$

Comparing this case to Section A.1.1, the only difference is in the $O(\varepsilon^0)$ order equation of motion

$$\rho^{\varepsilon}(\ddot{u}_i^0+u_{i,\tau\tau}^{2m})=\sigma_{ij,x_j}^0+\sigma_{ij,y_j}^1+b_i \tag{A.13}$$

where $\ddot{u}_i^0$ and θ^0 are both independent of the fast varying variables and represent the macroscopic displacement and temperature change; σ_{ij}^0 is defined in (A.4). Since $u_{i,\tau\tau}^{2m}$ vanishes due to the temporal averaging (A.2), and both σ_{ij,y_j}^m in (A.3) and σ_{ij,y_j}^1 in (A.13) have no contribution to the homogenized equation of motion due to local periodicity, the $O(\varepsilon^0)$ order homogenized equations in the present case are identical to those in Section A.1.1.

A.1.3. Mechanical fields: $\varepsilon_l=\varepsilon^m$, $\tau_0/t_r\geqslant 1$

The $O(\varepsilon^0)$ order equation of motion and energy equation can be expressed as

$$\begin{aligned}&\rho^{\varepsilon}\ddot{u}_i^0=\sigma_{ij,x_j}^0+\sigma_{ij,y_j}^m+b_i\\&\lambda^{\varepsilon}(\dot{\theta}^0+\theta^1_{,\tau})=q_{i,x_i}^0+q_{i,y_i}^1+V_{ijkl}^{\varepsilon}\dot{e}_{ij}^0\dot{e}_{kl}^0-T_0\beta_{ij}^{\varepsilon}\dot{e}_{ij}^0\end{aligned} \tag{A.14}$$

where $\ddot{u}_i^0$ and θ^0 are both independent of the fast varying variables and represent the macroscopic displacement and temperature change; the $O(\varepsilon^0)$ order stress tensor σ_{ij}^0 and strain rate tensor $\dot{e}_{kl}^0$ take the following form

$$\begin{aligned}&\sigma_{ij}^0=\left(L_{ijkl}^{\varepsilon}+V_{ijkl}^{\varepsilon}\frac{\partial}{\partial t}\right)(e_{klx}^0+e_{kly}^m)-\beta_{ij}^{\varepsilon}\theta^0\\&\dot{e}_{kl}^0=e_{klx,t}^0+e_{kly,\tau}^m\end{aligned} \tag{A.15}$$

where e_{kly}^m is determined by a local initial-boundary problem

$$\left\{\left(L_{ijkl}^{\varepsilon}+V_{ijkl}^{\varepsilon}\frac{\partial}{\partial t}\right)(e_{klx}^0+e_{kly}^m)-\beta_{ij}^{\varepsilon}\theta^0\right\}_{,y_j}=0 \tag{A.16}$$

(A.16) is in the similar form as those obtained by spatial homogenization (Francfort and Suquet, 1986; Sanchez-Palencia, 1980; Tartar, 1990). For the $O(\varepsilon^0)$ energy equation, the heat flux q_i^0 is determined by (A.11). The homogenized energy equation has the same form as (A.12), but the definition of $\dot{e}_{ij}^0$ follows (A.15). In this case, it is shown that the homogenized solutions obtained in Section 2.3.4 are not valid.

A.2. Thermal fields: $\varepsilon_l = \varepsilon$, $\varepsilon_\tau = \varepsilon^n$

A.2.1. Mechanical fields: $\varepsilon_l = \varepsilon^m$, $\varepsilon_\tau = \varepsilon$

In this case, the $O(\varepsilon^0)$ order equation of motion and its homogenized solutions are the same as those in Section A.1.1. For the $O(\varepsilon^0)$ energy equation,

$$\lambda^\varepsilon(\dot{\theta}^0 + \theta^n_{,\tau}) = q^0_{i,x_i} + q^1_{i,y_i} + V^\varepsilon_{ijkl}\dot{e}^0_{ij}\dot{e}^0_{kl} - T_0\beta^\varepsilon_{ij}\dot{e}^0_{ij} \tag{A.17}$$

where strain rate $\dot{e}^0_{ij}$ is given in (A.4); the heat flux q^0_i is determined by

$$q^0_i = k^\varepsilon_{ij}(\theta^0_{,x_j} + \theta^1_{,y_j}) \tag{A.18}$$

and $\theta^1_{,y_j}$ can be obtained by solving a linear unit cell problem

$$\left\{k^\varepsilon_{ij}(\theta^0_{,x_j} + \theta^1_{,y_j})\right\}_{,y_i} = 0 \tag{A.19}$$

Following (41)–(44), we have

$$\theta^1_{,y_j}(\boldsymbol{x}, \boldsymbol{y}, t) = \mu_{i,y_j}(\boldsymbol{y})\theta^0_{,x_i} \quad \text{and} \quad q^0_i = k^\varepsilon_{ij}(\delta_{jk} + \mu_{k,y_j}(\boldsymbol{y}))\theta^0_{,x_k} \tag{A.20}$$

where $\mu_i(\boldsymbol{y})$ is defined in (43). The homogenized energy equation is obtained by applying the spatial and temporal averaging processes to (A.17) and making use of (A.20) so that

$$\bar{\lambda}\dot{\theta}^0 = (\bar{k}_{ij}\theta^0_{,x_j})_{,x_i} + Q + \langle V^\varepsilon_{ijkl}\dot{e}^0_{ij}\dot{e}^0_{kl}\rangle - T_0\langle\beta^\varepsilon_{ij}\dot{e}^0_{ij}\rangle \tag{A.21}$$

where all the quantities have the same definitions as those in (64).

A.2.2. Mechanical fields: $\varepsilon_l = \varepsilon$, $\varepsilon_\tau = \varepsilon^m$

In this case, the $O(\varepsilon^0)$ order equation of motion and its homogenized solutions are the same as those in Section A.1.2, while the $O(\varepsilon^0)$ order energy equation and its homogenized solutions are the same as those in Section A.2.1.

A.2.3. Mechanical fields: $\varepsilon_l = \varepsilon^m$, $\tau_0/t_r \geqslant 1$

The homogenized equation of motion and energy equation in this case are the same as those obtained in Section A.1.3. Our solutions in Section 2.3.4 are not valid in this case.

A.3. Thermal fields: $\varepsilon_l = \varepsilon^n$, $\tau_0/t_\mathrm{r} \geqslant 1$

A.3.1. Mechanical fields: $\varepsilon_l = \varepsilon^m$, $\varepsilon_\tau = \varepsilon$

In this case the $O(\varepsilon^0)$ order equation of motion and its homogenized solutions are the same as those obtained in Section A.1.1. The $O(\varepsilon^0)$ order energy equation can be obtained by removing $\theta^1_{,\tau}$ in (A.5) and the homogenized energy equation is the same as that in Section A.1.1 since $\theta^1_{,\tau}$ vanishes due to temporal averaging.

A.3.2. Mechanical fields: $\varepsilon_l = \varepsilon$, $\varepsilon_\tau = \varepsilon^m$

In this case the $O(\varepsilon^0)$ order equation of motion and its homogenized solutions are the same as those obtained in Section A.1.2. The $O(\varepsilon^0)$ order energy equation can be obtained by removing $\theta^1_{,\tau}$ in (A.5) and the homogenized energy equation is the same as that in Section A.1.1.

A.3.3. Mechanical fields: $\varepsilon_l = \varepsilon^m$, $\tau_0/t_r \geqslant 1$

Similar to Section A.2.3, our homogenized solutions in Section 2.3.4 are not valid in this case.

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